Semantic Models of Quantum Programming Languages:

Recursion in Categorical Models

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Workshop on Higher Category Approach to Certifiably
Correct Quantum Information Processing Systems
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2015 MURI Project
Semantics and Tools for High Level Functional Quantum Programming Languages

- *Proto-Quipper* family of languages
  - Peter Selinger, Dalhousie[1]
- Coq verification of *Qwire*
  - Steve Zdancewic (UPenn)
- Structure of qubits / Quantum information
  - Patrick Hayden (Stanford)
- Recursion in Linear / Nonlinear Models[2] and Contextuality
  - Tulane component
- Dependent types
  - Aaron Stump (UIowa)
- Hoare logic and quantum languages
  - Xiaodi Wu (QICS, UMd)

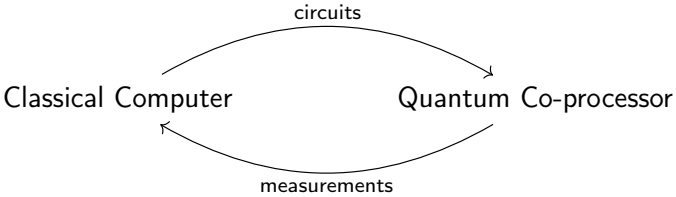
[1] Funded by other sources.

[2] This talk

Tulane University · The University of Iowa · Penn University of Pennsylvania · Stanford · Joint Center for Quantum Information and Computer Science

Prototypical Quantum Computer

- A *quantum computer* is a classical computer with a quantum co-processor



- Circuit: sequence of unitary operators

Prototypical Quantum Computer

- We elide measurements and focus on a classical functional language for *constructing circuits* and a linear language for *modeling them* as linear morphisms.
- A *quantum programming language* is a classical functional language together with a linear language of *quantum circuits*:

$$\begin{tikzcd} \textit{Functional Language} \arrow[r, bend left] & \textit{Linear language} \arrow[l, bend left] \end{tikzcd}$$

- We model *circuit description languages* using Linear / Nonlinear Models

Models of Functional Programming Languages

Classical Functional Language

$$\text{Types} \quad Q, R \quad ::= \quad \text{Bool} \mid \text{Nat} \mid Q + R \mid Q \times R \mid Q \to R$$

Based on intuitionistic logic and typed lambda calculus

Theorem [Lambek] There's a one-to-one correspondence between models of the typed lambda calculus and Cartesian closed categories

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Linear Functional Language

Types $\quad A, B \quad ::= \quad 0 \mid \mathsf{I} \mid A + B \mid A \otimes B \mid A \multimap B \mid !A$

Based on intuitionistic linear linear logic and linear lambda calculus

Models use symmetric monoidal closed categories.

Linear/Non-Linear models

A Linear/Non-Linear (LNL) model is given by the following data:

- A cartesian closed category $\mathbf{C}$.
- A symmetric monoidal closed category $\mathbf{L}$.
- A symmetric monoidal adjunction:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{L} \arrow[l, bend left, "G"] \end{tikzcd}$$

$$\begin{aligned} F(X \times Y) &\simeq F(X) \otimes F(Y) \\ F(X + Y) &\simeq F(X) + F(Y) \\ F(\emptyset) = 0 & \quad\;\; F(1) = I \\ F \circ G &= \; ! - \text{the lift comonad} \end{aligned}$$

An LNL model is a model of Intuitionistic Linear Logic.[1]

[1] Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models.* CSL'94

Proto-Quipper-M (*Rios and Selinger*)

Types	A, B	$::=$	$\alpha \mid 0 \mid A + B \mid \mathsf{I} \mid A \otimes B \mid A \multimap B \mid !A \mid \mathsf{Circ}(T, U)$
Intuitionistic types	P, R	$::=$	$0 \mid P + R \mid \mathsf{I} \mid P \otimes R \mid !A \mid \mathsf{Circ}(T, U)$
M-types	T, U	$::=$	$\alpha \mid \mathsf{I} \mid T \otimes U$

$$
\begin{array}{lll}
\text{Terms} & M, N & ::= \; x \mid \ell \mid c \mid \mathsf{let}\ x = M\ \mathsf{in}\ N \\
& & \mid \Box_A M \mid \mathsf{left}_{A,B} M \mid \mathsf{right}_{A,B} M \mid \mathsf{case}\ M\ \mathsf{of}\ \{\mathsf{left}\ x \to N \mid \mathsf{right}\ y \to P\} \\
& & \mid * \mid M; N \mid \langle M, N \rangle \mid \mathsf{let}\ \langle x, y \rangle = M\ \mathsf{in}\ N \mid \lambda x^A.M \mid MN \\
& & \mid \mathsf{lift}\ M \mid \mathsf{force}\ M \mid \mathbf{box}_T M \mid \mathsf{apply}(M, N) \mid (\vec{\ell}, C, \vec{\ell'})
\end{array}
$$

- All types other than Intuitionistic types are *linear*
- M-types: morphisms from a symmetric monoidal category such as M = FdC*Alg
- Only use one (combined) form of type judgement

Example

Assume $H : Q \multimap Q$ is a constant representing the Hadamard gate.

Example

two-hadamard : $\mathrm{Circ}(Q, Q)$
two-hadamard $\equiv \mathrm{box}_Q$ lift $\lambda q^Q.HHq$

This program creates a completed circuit consisting of two H gates. The term is intuitionistic (can be copied, deleted).

Circuit Model

Example

Shor's algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factoring an n-bit integer, for a fixed n.

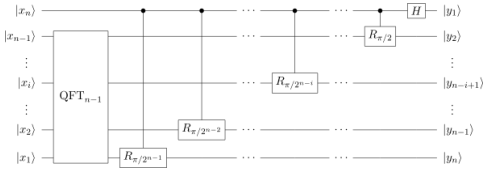

Figure: Quantum Fourier Transform on n qubits (subroutine in Shor's algorithm).[2]

Proto-Quipper-M is used to describe *families* of morphisms in an arbitrary, but fixed, symmetric monoidal category, **M**.

[2] Figure source: `https://commons.wikimedia.org/w/index.php?curid=14545612`

Concrete model of Proto-Quipper-M

A simple Proto-Quipper-M model is given by the LNL model:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "- \odot I"] \arrow[r, phantom, "\perp"] & \overline{\mathbf{M}} \arrow[l, bend left, "\overline{\mathbf{M}}(I, -)"] \end{tikzcd}$$

where $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{Set}]$ is a closed, product complete category containing given SMC $\mathbf{M}$

Theorem (Rios & Selinger)

The simple categorical model of Proto-Quipper-M is type-safe, sound, and computationally adequate

Concrete model of Proto-Quipper-M

There are two semantic models:

- For all types, $[\![P]\!] \in \overline{\mathbf{M}}$
- For intuitionistic types, also have $(\!|P|\!) \in \mathbf{Set}$

Theorem

For any intuitionistic type P, there exists a canonical isomorphism $\alpha_P : [\![P]\!] \to F(\!|P|\!)$.

So we can define copy and discard morphisms for each intuitionistic type P:

$$\Delta_P := [\![P]\!] \xrightarrow{\alpha_P} F(\!|P|\!) \xrightarrow{F\langle id, id\rangle} F((\!|P|\!) \times (\!|P|\!)) \xrightarrow{\cong} F(\!|P|\!) \otimes F(\!|P|\!) \xrightarrow{\alpha_P^{-1} \otimes \alpha_P^{-1}} [\![P]\!] \otimes [\![P]\!]$$

$$\diamond_P := [\![P]\!] \xrightarrow{\alpha_P} F(\!|P|\!) \xrightarrow{F1} F1 \xrightarrow{\cong} I$$

where $FX = X \odot I$

Our Work: Adding Recursion

- Focus on adding recursive *types*.
  - Term recursion follows from recursive types.
- Main difficulty is with the categorical model.
- How can we copy/discard intuitionistic recursive types?
  - A list of qubits should be *linear* – cannot copy/discard.
  - A list of natural numbers should be *intuitionistic* – can *implicitly* copy/discard.
- For the rest of the talk we focus on the linear/non-linear type structure.
- How do we design a linear/non-linear FPC[3] ?

[3] FPC is an intuitionistic type theory studied by Fiore and Plotkin.

Adding Recursive Types

Type Variables	X, Y		
Types	A, B	$::=$	$X \mid \alpha \mid A + B \mid \mathsf{I} \mid A \otimes B \mid A \multimap B \mid !A \mid \mathsf{Circ}(T, U)$ $\mid \mu X.A$
Intuitionistic types	P, R	$::=$	$X \mid P + R \mid \mathsf{I} \mid P \otimes R \mid !A \mid \mathsf{Circ}(T, U) \mid \mu X.P$
M-types	T, U	$::=$	$\alpha \mid \mathsf{I} \mid T \otimes U$

These types are accompanied by some formation rules, which we omit.

Design Choice: Two kinds of type variables – intuitionistic and linear? Or just one kind (as above)?

Some useful recursive types

Example

$\mathsf{Nat} \equiv \mu X.I + X \qquad$ (intuitionistic)

Example

$\mathsf{List\ Nat} \equiv \mu X.I + X \otimes \mathsf{Nat} \qquad$ (intuitionistic)

Example

$\mathsf{List\ Qubit} \equiv \mu X.I + X \otimes \mathsf{Qubit} \qquad$ (linear)

A **CPO**-enriched model

CPO – ω-complete partial orders and monotone maps preserving suprema of ω-chains.

A **CPO**–enriched LNL model includes:

1. A **CPO**-symmetric monoidal closed category $\mathcal{L}$ with finite **CPO**-coproducts.
2. A **CPO**-symmetric monoidal adjunction:

$$\begin{tikzcd} \mathbf{CPO} \arrow[rr, bend left, "F = -\odot I"] \arrow[rr, phantom, "\perp"] & & \mathcal{L}, \arrow[ll, bend left, "\mathcal{L}(I,-)"] \end{tikzcd}$$

3. The category $\mathcal{L}$ is $\mathbf{CPO}_{\perp!}$-enriched and has ω-colimits

Remark

1. and 3. imply $\mathcal{L}$ has a zero object and we can solve recursive domain equations.

Interpretation of recursive types

Interpreting recursive types requires finding initial (final) (co)algebras of certain **CPO**-endofunctors.

Lemma (Adámek)

Let $\mathbf{C}$ *be a category with an initial object* $\emptyset$ *and let* $T : \mathbf{C} \to \mathbf{C}$ *be an endofunctor. Assume further that the following* ω*-diagram*

$$\emptyset \xrightarrow{\iota} T\emptyset \xrightarrow{T\iota} T^2\emptyset \xrightarrow{T^2\iota} \cdots$$

has a colimit and T *preserves it. Then, the induced isomorphism is the initial* T*-algebra.*

Corollary

In a symmetric monoidal closed category with finite coproducts and ω*-colimits, any endofunctor composed from constants,* $\otimes$ *and* $+$ *has an initial algebra.*

Embedding-projection pairs

Problem: How do we interpret recursive types which also contain ! and $\multimap$?
Textbook Solution: **CPO**-enrichment and embedding-projection pairs.

Definition

Given a **CPO**-enriched category $\mathbf{C}$, an *embedding-projection* pair is a pair of morphisms $e : A \to B$ and $p : B \to A$, such that $p \circ e = \mathrm{id}$ and $e \circ p \leq \mathrm{id}$.

Theorem

If e is an embedding, then it has a unique projection, which we denote e^*.

Definition

The subcategory of $\mathbf{C}$ with the same objects, but whose morphisms are embeddings is denoted $\mathbf{C}_e$.

Interpretation of recursive types (contd.)

Theorem (Smyth and Plotkin)

*If $T : \mathbf{C} \to \mathbf{D}$ is a **CPO**-enriched functor and $\mathbf{C}$ has ω-colimits, then T preserves ω-colimits of embeddings. In other words, the restriction $T_e : \mathbf{C}_e \to \mathbf{D}_e$ is ω-continuous.*

Theorem

*In our categorical model, any **CPO**-endofunctor $T : \mathcal{L} \to \mathcal{L}$ has an initial T-algebra, whose inverse is a final T-coalgebra.*

Remark

The above theorem follows directly from results in Fiore's PhD thesis.

Main Lemma

We define $\mathbf{CPO}_{pe}$ to be the full-on-objects subcategory of $\mathbf{CPO}$ whose morphisms f are those satisfying $F(f) \in \mathcal{L}_e$. We call such f *pre-embeddings*.

Then there are two semantic models:

- For all types, $[\![\Theta \vdash P]\!] \in \mathcal{L}$
- For intuitionistic types, also have $(\!|\Theta \vdash P|\!) \in \mathbf{CPO}_{pe}$

There exists a natural isomorphism

$$\alpha_{\Theta \vdash P} : [\![\Theta \vdash P]\!]_s \circ F^{\times n} \Longrightarrow F \circ (\!|\Theta \vdash P|\!)$$

Diagrammatically:

$$\begin{tikzcd}
\mathcal{L}_e^{|\Theta|} \arrow[rr, "{[\![\Theta \vdash P]\!]_e}"] & \arrow[d, phantom, "\overset{\alpha}{\Nearrow}"] & \mathcal{L}_e \\
\mathbf{CPO}_{pe}^{|\Theta|} \arrow[u, "F^{\times|\Theta|}"] \arrow[rr, "{(\!|\Theta \vdash P|\!)}"'] & & \mathbf{CPO}_{pe} \arrow[u, "F"']
\end{tikzcd}$$

Copy and Discard

Let P be an intuitionistic object and $\alpha : P \to F(X)$ an isomorphism.

We can define three maps:

Discard: $\diamond_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(1_X)} F(1) \xrightarrow{\cong} I$;

Copy: $\Delta_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\langle \mathrm{id}, \mathrm{id} \rangle)} F(X \times X) \xrightarrow{\cong} F(X) \otimes F(X) \xrightarrow{\alpha^{-1} \otimes \alpha^{-1}} P \otimes P$;

Lift: $\mathbf{lift}_P^\alpha := P \xrightarrow{\alpha} F(X) \xrightarrow{F(\eta_X)} !F(X) \xrightarrow{!(\alpha^{-1})} !P$.

Given two intuitionistic objects P_1 and P_2, a morphism $f : P_1 \to P_2$ is called *intuitionistic*, if there exists a morphism $f' \in \mathbf{CPO}(X, Y)$ and two isomorphisms

α and β, such that $f = P_1 \xrightarrow{\alpha} F(X) \xrightarrow{F(f')} F(Y) \xrightarrow{\beta} P_2$.

If $f : P_1 \to P_2$ is intuitionistic, then:

- $\diamond_{P_2} \circ f = \diamond_{P_1}$;
- $\Delta_{P_2} \circ f = (f \otimes f) \circ \Delta_{P_1}$;
- $\mathbf{lift}_{P_2} \circ f = !f \circ \mathbf{lift}_{P_1}$.

Thank You!

Questions??

Syntax

$$\frac{}{\Phi, x : A; \emptyset \vdash x : A}\ (\text{var}) \qquad \frac{}{\Phi; \ell : \alpha \vdash \ell : \alpha}\ (\text{label}) \qquad \frac{}{\Phi; \emptyset \vdash c : A_c}\ (\text{const}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \qquad \Phi, \Gamma_2, x : A; Q_2 \vdash n : B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } x = m \text{ in } n : B}\ (\text{let})$$

$$\frac{\Gamma; Q \vdash m : 0}{\Gamma; Q \vdash \square_C m : C}\ (\text{initial}) \qquad \frac{\Gamma; Q \vdash m : A}{\Gamma; Q \vdash \text{left}_{A,B} m : A + B}\ (\text{left}) \qquad \frac{\Gamma; Q \vdash m : B}{\Gamma; Q \vdash \text{right}_{A,B} m : A + B}\ (\text{right}) \qquad \frac{}{\Phi; \emptyset \vdash * : I}\ (*)$$

$$\frac{\Phi, \Gamma_1; Q_1 \vdash m : A + B \qquad \Phi, \Gamma_2, x : A; Q_2 \vdash n : C \qquad \Phi, \Gamma_2, y : B; Q_2 \vdash p : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{case } m \text{ of } \{\text{left } x \to n \mid \text{right } y \to p\} : C}\ (\text{case}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : I \qquad \Phi, \Gamma_2; Q_2 \vdash n : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash m; n : C}\ (\text{seq})$$

$$\frac{\Phi, \Gamma_1; Q_1 \vdash m : A \qquad \Phi, \Gamma_2; Q_2 \vdash n : B}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \langle m, n \rangle : A \otimes B}\ (\text{pair}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \otimes B \qquad \Phi, \Gamma_2, x : A, y : B; Q_2 \vdash n : C}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{let } \langle x, y \rangle = m \text{ in } n : C}\ (\text{let-pair})$$

$$\frac{\Gamma, x : A; Q \vdash m : B}{\Gamma; Q \vdash \lambda x^A.m : A \multimap B}\ (\text{abs}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : A \multimap B \qquad \Phi, \Gamma_2; Q_2 \vdash n : A}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash mn : B}\ (\text{app}) \qquad \frac{\Phi; \emptyset \vdash m : A}{\Phi; \emptyset \vdash \text{lift } m : !A}\ (\text{lift}) \qquad \frac{\Gamma; Q \vdash m : !A}{\Gamma; Q \vdash \text{force } m : A}\ (\text{force})$$

$$\frac{\Gamma; Q \vdash m : !(T \multimap U)}{\Gamma; Q \vdash \text{box}_T m : \text{Diag}(T, U)}\ (\text{box}) \qquad \frac{\Phi, \Gamma_1; Q_1 \vdash m : \text{Diag}(T, U) \quad \Phi, \Gamma_2; Q_2 \vdash n : T}{\Phi, \Gamma_1, \Gamma_2; Q_1, Q_2 \vdash \text{apply}(m, n) : U}\ (\text{apply}) \qquad \frac{\emptyset; Q \vdash \vec{\ell} : T \quad \emptyset; Q' \vdash \vec{\ell'} : U \quad S \in \mathbf{M}_L(Q, Q')}{\Phi; \emptyset \vdash (\vec{\ell}, S, \vec{\ell'}) : \text{Diag}(T, U)}\ (\text{diag})$$

Operational semantics

$$\frac{(S, m) \Downarrow (S', v) \quad (S', n) \Downarrow (S'', v')}{(S, \langle m, n \rangle) \Downarrow (S'', \langle v, v' \rangle)} \qquad \frac{(S, m) \Downarrow (S', \langle v, v' \rangle) \quad (S', n[v \,/\, x, v' \,/\, y]) \Downarrow (S'', w)}{(S, \mathrm{let}\ \langle x, y \rangle = m\ \mathrm{in}\ n) \Downarrow (S'', w)}$$

$$\frac{}{(S, \mathrm{lift}\ m) \Downarrow (S, \mathrm{lift}\ m)} \qquad \frac{(S, m) \Downarrow (S', \mathrm{lift}\ m') \quad (S', m') \Downarrow (S'', v)}{(S, \mathrm{force}\ m) \Downarrow (S'', v)}$$

$$\frac{(S, m) \Downarrow (S', \mathrm{lift}\ n) \quad \mathrm{freshlabels}(T) = (Q, \vec{\ell}) \quad (\mathrm{id}_Q, n\vec{\ell}) \Downarrow (D, \vec{\ell}')}{(S, \mathrm{box}_T\, m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}'))}$$

$$\frac{(S, m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}')) \quad (S', n) \Downarrow (S'', \vec{\kappa}) \quad \mathrm{append}(S'', \vec{\kappa}, \vec{\ell}, D, \vec{\ell}') = (S''', \vec{\kappa}')}{(S, \mathrm{apply}(m, n)) \Downarrow (S''', \vec{\kappa}')}$$

$$\frac{(S, m) \Downarrow (S', (\vec{\ell}, D, \vec{\ell}')) \quad (S', n) \Downarrow (S'', \vec{\kappa}) \quad \mathrm{append}(S'', \vec{\kappa}, \vec{\ell}, D, \vec{\ell}')\ \mathrm{undefined}}{(S, \mathrm{apply}(m, n)) \Downarrow \mathrm{Error}} \qquad \frac{}{(S, (\vec{\ell}, D, \vec{\ell}')) \Downarrow (S, (\vec{\ell}, D, \vec{\ell}'))}$$