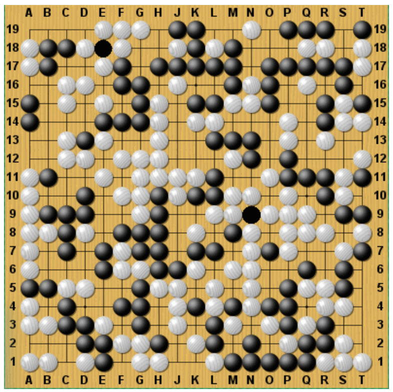
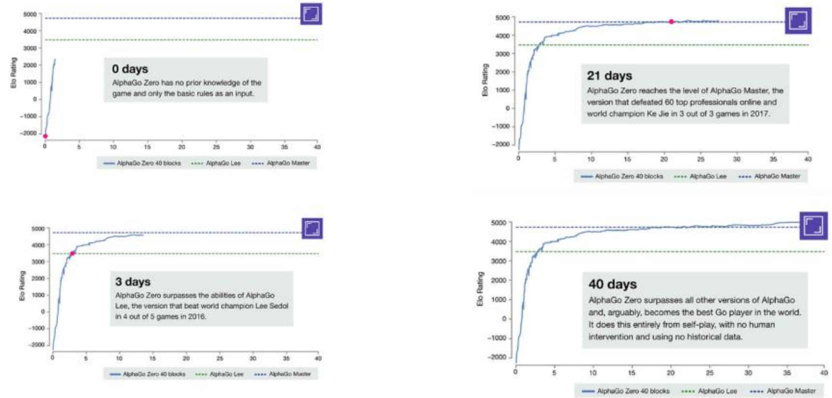
Is the Quantum Model of Computing Unique?

or Simulation of Physics by Quantum Computers



John Tromp



Arpad Elo

Zhenghan Wang

Microsoft Station Q & UC Santa Barbara

DC, Feb.4, 2019

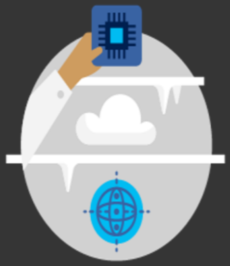
Microsoft's unique approach

Revolutionary topological approach

Our quantum approach brings theory to reality, harnessing topological qubits that perform computations longer and more consistently, with fewer errors.

Bold investments and a global team

For more than a decade, we've made consistent investments and built the quantum dream team with collaboration across universities, industries, and more.



Scalable, end-to-end technology

Our full-stack quantum-computing solution is designed so you and your developers can approach quantum computing right away, with the ability to scale.

How Microsoft’s TQFT Computer Is Doing?

TQC (topological) vs tQC (traditional)

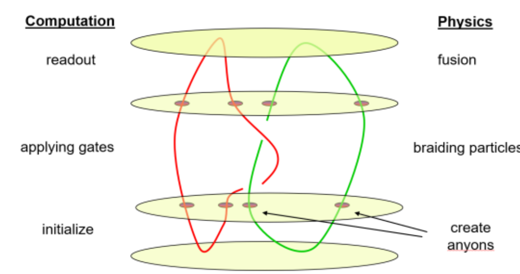
- Theoretically ground states of **topological phases of matter (where information stored)** form error correction codes
- **No** fully controlled **topological qubits** yet

	TQC	tQC
$Physical$	NA	~ 72
$Logical$	$\epsilon < 1$	$\delta < 1$

Freedman Kitaev

Topological Quantum Computation (TQC)

Freedman 97, Kitaev 97, Freedman-Larsen-W. 00



- Mathematics of topological quantum computing (with E. Rowell), Bull. AMS, vol. 55, 2018, arXiv:1705.06206
- Topological quantum computation, CBMS monograph, vol. 112, 2010, http://web.math.ucsb.edu/~zhenghwa/course.php

Algorithm

- **Hilbert 1928---Entscheidungsproblem (the decision problem)**
- **Turing 1936---No (On computable numbers)**
- **Church-Turing Thesis:** Any reasonable computation can be done by Turing machines. Algorithm=TM

  Turing machine can simulate any physical system, but slowly.

  So assume any physical theory is computable.

- **Extended** Church-Turing Thesis: **Efficiently**

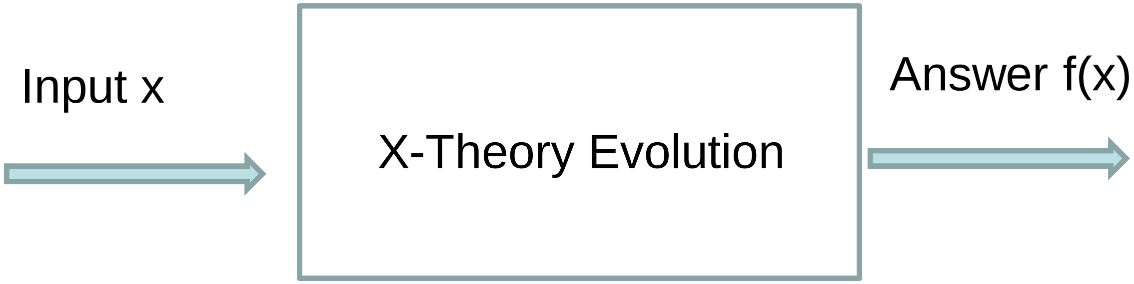
  All physical theories can be simulated by TMs efficiently.

  Potential counterexample: Quantum computing

- **Quantum** Church-Turing thesis:

There is a unique quantum model of computing or a physical quantum field theory can be **efficiently** simulated by quantum computers.

Computational Power of Physical Theories



Each theory provides a computational model,
which selects class of efficiently solvable problems XP

"As a generality, we propose that each physical theory supports computational models whose power is limited by the physical theory. It is well known that classical physics supports a multitude of the implementation of the Turing machine". Freedman further suggested that computational models based on some TQFTs might be more powerful than quantum computing—the computing model based on quantum mechanics. But when accuracy and measurement are carefully analyzed, the computing model based on TQFTs are polynomially equivalent to quantum computing

P/NP, and the quantum field computer

MICHAEL H. FREEDMAN

Classical Physics	**Turing Machines P**
Quantum Mechanics	**Quantum Circuit Model BQP**
Quantum Field Theory	**? BQP**
String Theory	**??? BSP**

Rigorous QFTs:
TQFTs, CFTs, …

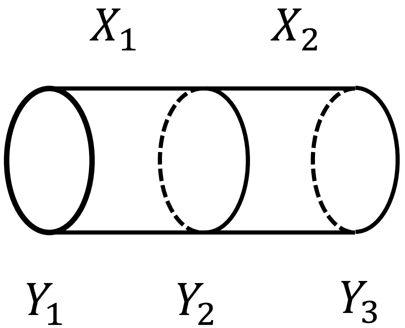
True for TQFTs (Freedman, Kitaev, W. 02), i.e. BQP=B-TQFT-P
CFTs?

Quantum Field Theories

- Need a mathematical definition of QFTs
- **Know trivial QFTs=TQFTs**: T stands both for trivial and topological
- Also 1+1 conformal field theories (CFTs)
- A great opportunity again to examine mathematically what is QFT.

Atiyah-Segal (2+1)-TQFT

A symmetric monoidal “functor” (V, Z):
category of 2-3-mfds → Vec
2-mfd Y → vector space $V(Y)$
3-bord X from Y_1 to Y_2 → Z(X): $V(Y_1) \to V(Y_2)$



- $V(S^2) \cong \mathbb{C} \longrightarrow V(\emptyset) = \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I) = \mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup X_2) = \kappa^m \cdot Z(X_1) \cdot Z(X_2)$ (anomaly)

What to Simulate

- **Two things:**

  Evolutions and Partition functions

  Representations of mapping class groups (e.g. braid groups) and 3-manifold invariants

- **Localization of TQFTs:**

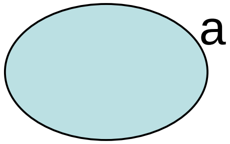
  Key is some form of gluing formulas---

  quantum state can be reconstructed from patches

Extended (2+1)-TQFT Axioms

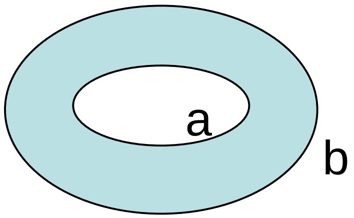
Moore-Seiberg, Walker, Turaev,…

Let L={a,b,c,…d} be the labels (particle types), a → a*, and a**=a, 0 (or 1) =trivial type



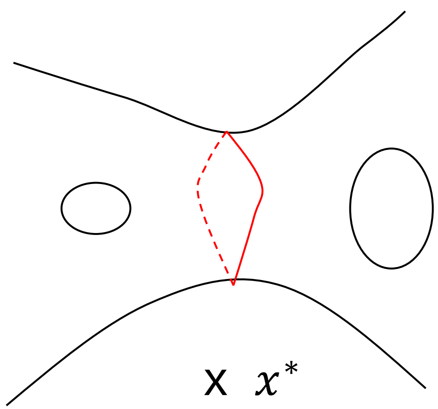
Disk Axiom:

$V(D^2; a)=0$ if $a\neq 0$, $\mathbb{C}$ if $a=0$



Annulus Axiom:

$V(A; a,b)=0$ if $a\neq b^*$, $\mathbb{C}$ if $a=b^*$



Gluing Axiom:

$V(Y; l) \cong \oplus_{x\varepsilon L} V(Y_{cut}; l,x,x^*)$

Qudit: $\boldsymbol{W = \oplus_{a,b,c} V_{a,b,c}}$

Simulating TQFTs (Freedman-Kitaev-W.)

Given some anyon statistics or rep of the braid groups

$$\rho: B_n \longrightarrow U(V_n),$$

Find a quantum circuit on m qudits

$$U_L: W^{\otimes m} \longrightarrow W^{\otimes m}$$

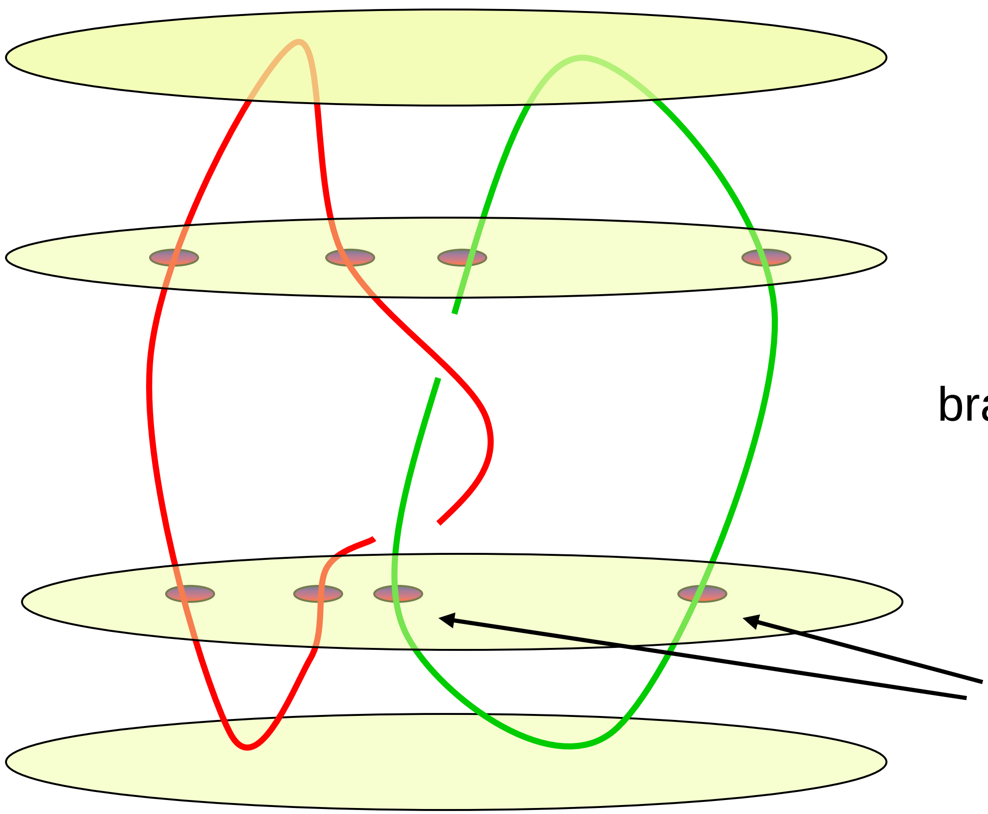
so that the following commutes:

$$\begin{array}{ccc} V_n & \rightarrow & W^{\otimes m} \\ \downarrow \rho & & \downarrow U_L \\ V_n & \rightarrow & W^{\otimes m} \end{array}$$

Simulation of TQC=Approximation Link Invariants

Freedman-Kitaev-W.

Computation		**Physics**
readout		fusion
applying gates		braiding particles
initialize		create anyons



CFTs as Boundaries of TQFTs

Edge physics of fractional quantum Hall liquids:

∂Witten-Chern-Simons theories

$\sim$Wen's chiral Luttinger liquids

∂TQFTs/UMTCs $\sim$ χCFTs/Chiral algebras

Can we extend the simulation to the boundaries?

Bulk-edge Connection of Topological Phases

- Edge physics of fractional quantum Hall liquids:

  ∂Witten-Chern-Simons theories$\sim$Wen's chiral Luttinger liquids

  ∂TQFTs/UMTCs $\sim$ χCFTs/Chiral algebras

- Tannaka-Krein duality (Gannon):

  Reconstruct chiral algebras from UMTCs=Rep(Chiral algebras)

  Symmetric fusion categories are 1-1 correspondence with pairs (G, μ)

Chiral and Full Conformal Field Theories

The BPZ definition of conformal field theory is that it is an inner product space $\mathscr{H}$ which can be decomposed into a direct sum

$$\mathscr{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c}) \tag{2.1}$$

of irreducible highest weight modules of $\hat{Vir}_c \times \overline{\hat{Vir}}_{\bar{c}}$ such that

1. There is a unique $SL_2(R) \times SL_2(R)$ invariant states $|0\rangle$ with $(h,\bar{h})=(0,0)$.
2. For each vector $\alpha \in \mathscr{H}$ there is an operator $\phi_\alpha(z)$ on $\mathscr{H}$, parametrized by $z \in C$. Also, for every operator ϕ_α there exists a conjugate operator ϕ_{α^*} (partially) characterized by the requirement that the operator product expansion $\phi_\alpha \phi_{\alpha^*}$ contains a descendant of the unit operator.
3. For $\alpha = i$ a highest weight state we have $[L_n, \phi_i(z,\bar{z})] = \left(z^{n+1} \frac{d}{dz} + \Delta_i (n+1) z^n \right) \phi_i$.
4. The inner products $\langle 0 | \phi_{i_1}(z_{i_1}, \bar{z}_{i_1}) \dots \phi_{i_n}(z_{i_n}, \bar{z}_{i_n}) |0\rangle$ exist for $|z_{i_1}| > \dots |z_{i_n}| > 0$ and admit an unambiguous real-analytic continuation, independent of ordering, to C^n minus diagonals. This is called the assumption of duality.
5. The one-loop partition function and correlation functions, computed as traces exist and are modular invariant.

We now discuss the notion of chiral algebras, or vertex algebras [26]. The fields in a conformal field theory form a closed operator product expansion. An important subset of the fields are the holomorphic fields. Since the operator product expansion of two holomorphic fields is holomorphic, these form a closed subalgebra of the operator product algebra called the "chiral algebra," $\mathscr{A}$, of the theory.[2] Every conformal field theory has at least two holomorphic fields given by the unit operator and the stress tensor: 1, $T(z)$ and thus every chiral algebra contains the (enveloping algebra of the) Virasoro algebra. We can choose a basis $\{\mathcal{O}^i(z)\}$ for $\mathscr{A}$ such that each field has a well-defined dimension Δ_i. By the axiom of duality, fields in a conformal field theory have no relative monodromy, in particular, the weights Δ_i are integers. Defining modings $\mathcal{O}^i(z) = \sum_n \mathcal{O}^i_n z^{-n-\Delta_i}$ we can write the operator product algebra in two equivalent ways:

$$\begin{aligned} \mathcal{O}^i(z)\mathcal{O}^j(w) &= \sum_k \frac{c_{ijk}}{(z-w)^{\Delta_{ijk}}} \mathcal{O}^k(w), \\ [\mathcal{O}^i_n, \mathcal{O}^j_m] &= \sum_k c_{ijk}(n,m) \mathcal{O}^k_{n+m}, \end{aligned} \tag{2.2}$$

($\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k$). Using the moding one can define Verma modules and irreducible quotients and, therefore, one can speak of the irreducible representations $\mathscr{H}_i$ of $\mathscr{A}$.

Classical and quantum conformal field theory

G Moore, N Seiberg - Communications in Mathematical Physics, 1989

Chiral CFT (χCFT) = the chiral algebra mathematically=vertex operator algebra (VOA).
A full CFT is determined by a VOA V plus an indecomposable module category over Rep(V).
All CFTs are (1+1)- unitary rational ones.

Vertex Operator Algebra (VOA)

A VOA is a quadruple (V, Y, 1, ω), where $V = \bigoplus_n V_n$ is Z-graded vector space and

$$Y : V \to \mathfrak{F}(V),\ v \mapsto Y(v,z) = \sum v_n z^{-n-1}$$
$$\mathbf{1}, \omega \in V,\ \mathbf{1} \neq 0.$$

The fields $Y(v,z)$ are mutually local and creative, and the following hold:

$$Y(\omega, z) = \sum L_n z^{-n-2} \text{ with a constant } c \text{ such that}$$
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} c \mathrm{Id}_V$$
$$V_n = \{v \in V_n \mid L_0 v = nv\}$$
$$\dim V_n < \infty,\ V_n = 0 \text{ for } n \ll 0$$
$$Y(L_{-1}u, z) = \partial Y(u,z)$$

$$\textit{locality} : Y(u,z) \sim Y(v,z)$$
$$\textit{creativity} : Y(u,z)\mathbf{1} = u + O(z)$$

$\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field}\}.$

I.e. for $v \in V$ there is an integer N (depending on v) such that $a_n(v) = 0$ for all $n > N$.

a(z) is a field if it satisfied the truncation condition,

Some Issues

- Infinite dimensional Hilbert space
- No gluing formulas
- What does locality mean?
- …

One answer: Anyonic Chains

- Jones: a general spin chain should be tensored together by Connes fusion.
- Explicit Hamiltonian for golden chain by Station Q---exactly solvable, but not known rigorously solvable.

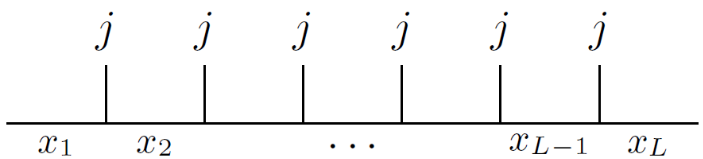
Ising Model

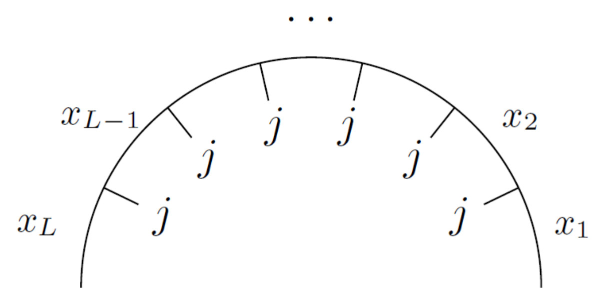
Anyon types: $\{1, \sigma, \psi\}$
Fusion rules: $\sigma^2 = 1 + \psi, \sigma\psi = \psi\sigma = \sigma, \psi^2 = 1$
Quantum dimensions: $\{1, \sqrt{2}, 1\}$
Twists: $\theta_1 = 1, \theta_\sigma = e^{\frac{\pi i}{8}}, \theta_\psi = -1$
Total quantum order: $D = 2$
Topological central charge: $c = \frac{1}{2}$
Braidings: $R_1^{\sigma\sigma} = e^{-\frac{\pi i}{8}}, R_1^{\psi\psi} = -1, R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = -i, R_\psi^{\sigma\sigma} = e^{\frac{3\pi i}{8}}$

S-matrix: $S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$

F-matrices: $F_\sigma^{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, F_\sigma^{\psi,\sigma,\psi} = (-1), F_\psi^{\sigma,\psi,\sigma} = (-1),$

Anyonic Chains





$$H = J \sum_j P_j^0,$$

$$H = -\frac{1}{d} \sum_{i=1}^{L-1} e_i.$$

Scaling limits of ACs of $spin\frac{1}{2}\ anyon\ in\ SU(2)_k$ = UMMs with central charge $c = 1 - \frac{6}{(k+1)(k+2)}$.

Emergence of Conformal Symmetry

- The Temperley-Lieb generators e_i are finite versions of the Virasoro generators L_n. How are they related precisely?
- Our conjectured formulas:

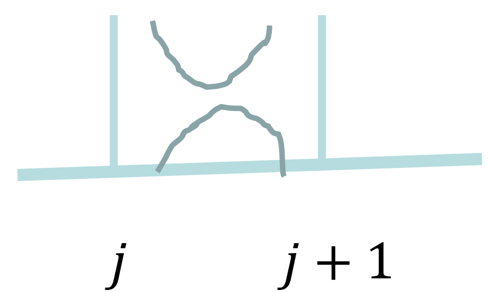
$$O_n^c = -\sum_{j=1}^{2n-1} \cos\left(\frac{m(j+\frac{1}{2})\pi}{2n+1}\right)e_j, \quad O_n^s = i\sum_{j=1}^{2n-2} \sin\left(\frac{m(j+1)\pi}{2n+1}\right)[e_j, e_{j+1}],$$

we have operators $\widetilde{L}^c_{\pm m}, \widetilde{L}^s_{\pm m} \xrightarrow{SL} L_{\pm m}$ *satisfying the properties in* $\boxed{\mathbf{4.3}}$ *and,*

$$\frac{\widetilde{L}^c_m + \widetilde{L}^c_{-m}}{2} = \alpha_n^c O_n^c + \beta_n^{m,c}\mathbf{1} \xrightarrow{SL} \frac{L_m + L_{-m}}{2},$$

$$\frac{i(\widetilde{L}^s_m - \widetilde{L}^s_{-m})}{2} = \alpha_n^s O_n^c + \beta_n^{m,s}\mathbf{1} \xrightarrow{SL} \frac{i(L_m - L_{-m})}{2},$$

where $\alpha_n^c, \alpha_n^s, \beta_n^{m,c}$, *and* $\beta_n^{m,s}$ *are suitable scaling factors.*



Comparison with Koo-Saleur formulas: $j+1 \; vs \; j+\frac{1}{2}$

- Heuristically:

$$e_\theta \xrightarrow{SL} Y(\omega, e^{i\theta})$$

Algorithmic Scaling Limit

Denote by $\mathcal{W}_n^M$ the Hilbert space $\mathcal{W}_n$ restricted to energies at most M, i.e. $\mathcal{W}_n^M = \bigoplus_{\lambda_i^{(n)} \leq M} E_{\lambda_i^{(n)}}$. Assume the following set of properties (**P**)

- $\lambda_i = \lim_{n\to\infty} \lambda_i^{(n)}$ exists for all $i \in \mathbb{N}$ with the convention $\lambda_i^{(n)} = 0$ for $i > d(n)$,
- (*connecting maps*) for all $M > \lambda_1$, there exist *connecting* unitary maps $\phi_n^M : \mathcal{W}_n^M \to \mathcal{W}_{n+1}^M$ for large enough n,
- (*extension*) ϕ_n^M is an extension of $\phi_n^{M'}$ when $M \geq M'$.

Scaling limit of observables as algebras.
Rate of convergence---scaling dimension of irrelevant operators.
…

The definition is inspired by earlier works of Read-Saleur and numerical simulation of anyonic chains.

Scaling Limit of Ising ACs

(a) $\mathcal{W}_n = (\frac{1}{2}, \frac{1}{2})$, $H_n = -\sum_{j=1}^{2n-1} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0 + \chi_{\frac{1}{2}}, L_0)$.

(b) $\mathcal{W}_n = (0,0)$ *or* $(1,1)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0, L_0)$.

(c) $\mathcal{W}_n = (0,1)$ *or* $(1,0)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{2}}, L_0)$.

(d) $\mathcal{W}_n = (\frac{1}{2}, 1)$ *or* $(\frac{1}{2}, 0)$, $H_n = -\sum_{j=1}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{16}}, L_0)$.

(e) $\mathcal{W}_n$ *be the periodic chain of size* $2n$, *and* $H_n = -\sum_{j=1}^{2n} e_j$. *Then*

$$(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}, L_0 + \overline{L}_0)$$

if n *is even.*

Furthermore, the rate of convergence of each scaling limit is $O(\frac{1}{n})$ *while we have restriction of energies up to* $O(\sqrt[3]{n})$.

2- For the corresponding higher Virasoro generators action, with the same rate of convergence as above, given a fixed $m \neq 0$, *we have (up to some scalings)*

(a) $-\sum_{j=1}^{2n-1} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1}) e_j \xrightarrow{SL} L_m + L_{-m}$,

$i\sum_{j=1}^{2n-2} \sin(\frac{m(j+1)\pi}{2n+1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

Compute Correlations and Evolutions

Approximate a suitable normalization with a given error:

$$|(1, \mathcal{Y}_n(a_n, z_n) \ldots \mathcal{Y}_1(a_1, z_1)1)|.$$

Simulate Unitary Evolutions:

$$L(f) = \sum_{|m| \leq N} \hat{f}_m L_m$$

Given a function f with $O(1)$ many coefficients which can be computed in $O(1)$ time, approximate the following up to an error ϵ with no more than polynomially many steps in $\frac{1}{\epsilon}$:

$$|(\Omega, e^{iL(f)}\Omega)|.$$

Rep of Diff(S^1)???

Happy Chinese New Year of Pig 🐖猪

- **Men**

Born in the Pig year are optimistic and gentle. They are very focused. Once they decide on a goal, they'll put everything into it.

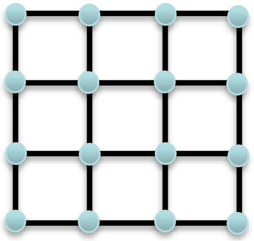
- **Women**

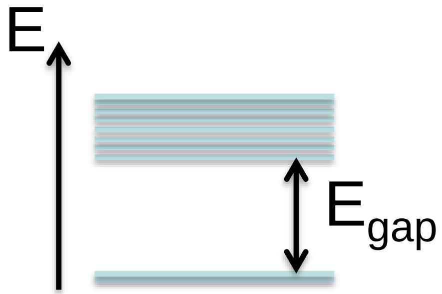
Born in the Pig year are full of excitement. They attend social events whenever possible and treat everyone genuinely. Combined with their easygoing personality, they gain everyone's trust.

Anyons: Topological Phases

Local Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$

Local, Gapped Hamiltonian $H : \mathcal{H} \to \mathcal{H}$



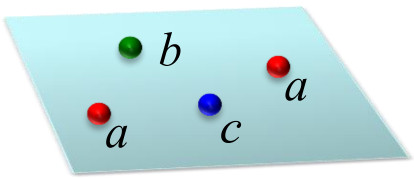


Two gapped Hamiltonians H_1, H_2 realize the same topological phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase is a class of gapped Hamiltonians that realize the same TQFT or anyon model in low energy.

Anyons: Quarsi-particles

Finite-energy topological quasiparticle excitations=anyons



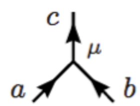
Quasiparticles a, b, c

Two quasiparticles have the same topological charge
or anyon type if they differ by local operators

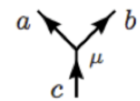
Anyons in 2+1 dimensions described mathematically by a
Unitary Modular Tensor Category $\mathcal{C}$

Anyon Model: Data

- Anyon types {a, b, c, …}
- Fusion Rules $a \times b = \sum_c N_{ab}^c c$
- Fusion/Splitting spaces:

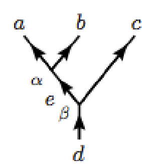


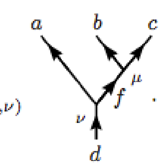
$\propto \langle a, b; c, \mu | \in V_{ab}^c$



$\propto | a, b; c, \mu \rangle \in V_c^{ab}$

- F-Symbols



$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$


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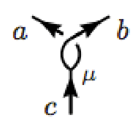
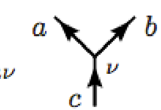
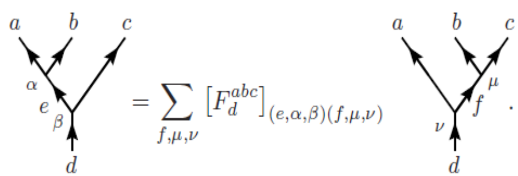
- Braiding (R-Symbols)



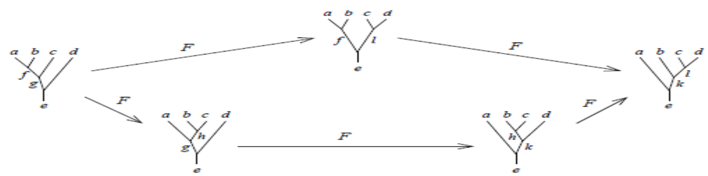
$= \sum_{\nu} \left[R_c^{ab}\right]_{\mu\nu}$


Anyons: Simple Objects in UMTC = Anyon Model

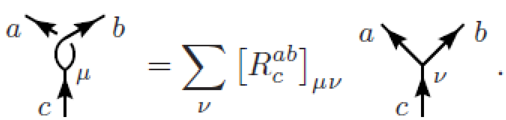
An anyon model: **a collection of numbers {L, N_{ab}^{c}, $F_{d;nm}^{abc}$, R_{c}^{ab}, ϵ_i} that satisfy some polynomial constraint equations including pentagons and hexagons.**



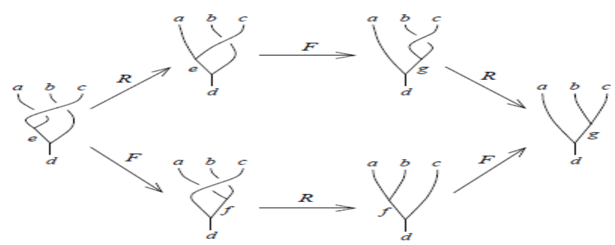
6j symbols for recoupling



Pentagons for 6j symbols



R-symbol for braiding



Hexagons for R-symbols

Majorana System=Ising Theory

Ising Anyon σ and Majorana fermion ψ

Anyon types: **{1, σ, ψ} or {0,1/2, 1}**

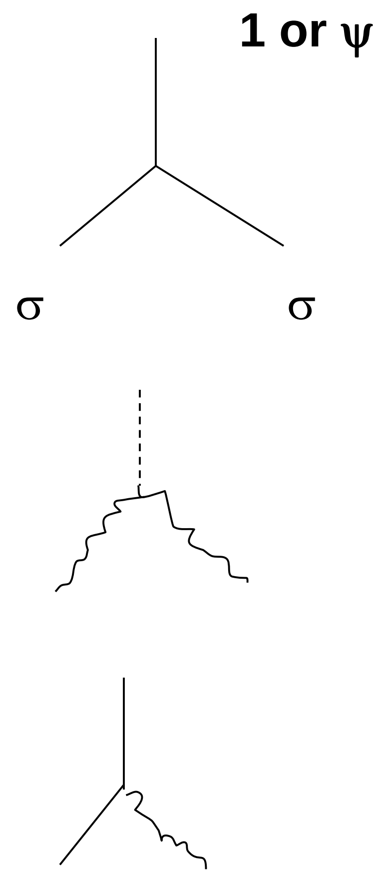
Quantum dim: $\{\mathbf{1}, \sqrt{\mathbf{2}}, \mathbf{1}\}$

Fusion rules:

$\sigma^2 \cong 1+\psi$,

$\psi^2 \cong 1$,

$\sigma\ \psi \cong \psi\ \sigma \cong \sigma$



1---ground state

ψ---self-conjugate (Majorana) fermion

σ---Ising anyon

All self-conjugate